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The quantum–classical crossover in the adiabatic response of chaotic systems

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Abstract. The autocorrelation function of the force acting on a slow classical system, resulting from interaction with a fast quantum system is calculated following Berry–Robbins, Wilkinson and Jarzynski within the leading order correction to the adiabatic approximation. The time integral of the autocorrelation function is often proportional to the rate of energy transfer between the systems. The fast quantum system is assumed to be chaotic in the classical limit for each configuration of the slow system. An analytic formula is obtained for the finite-time integral of the correlation function, in the framework of random matrix theory (RMT), for a specific dependence on the adiabatically varying parameter. Extension to a wider class of RMT models is discussed. For the Gaussian unitary and symplectic ensembles for long times the time integral of the correlation function vanishes or falls off as a Gaussian with a characteristic time that is proportional to the Heisenberg time, depending on the details of the model. The fall-off is inversely proportional to time for the Gaussian orthogonal ensemble. The correlation function is found to be dominated by the nearest-neighbour level spacings. It was calculated for a variety of nearest-neighbour level spacing distributions, including ones that do not originate from RMT ensembles. The various approximate formulae obtained are tested numerically in RMT. The results shed light on the quantum to classical crossover for chaotic systems. The implications on the possibility to experimentally observe deterministic friction are discussed.

1. Introduction

Dissipation of energy from a physical system to a thermal bath takes place as a result of a fluctuating force that acts on the system because of its coupling to the bath. The dissipative friction force is related to the correlation function of the fluctuating force. The force resulting from coupling to a chaotic system, rather than to a bath, is also fluctuating. The question that will be studied in this paper is on what timescales it leads to friction (or other energy transfer mechanisms between the systems) and what is the relation of this friction to the autocorrelation function of the fluctuating force. An example is a slow particle coupled to a fast particle so that the motion of the fast particle is chaotic for each position of the slow particle.

Various models for dissipation of energy from a slow particle by a fast one have been developed. To our knowledge the first models of this type were introduced in the context of nuclear physics [1]. In particular, a model where particles move within a region bounded by a deforming boundary, modelling the nuclear surface was studied [2]. The energy transferred between the boundary and particles enclosed inside was calculated classically and quantum mechanically in the framework of some approximations. Recently some detailed

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numerical simulations were performed along these lines and the regime of validity of various approximations was tested [3].

Quantum dissipation for a wide class of model systems was explored in several studies [4–9]. Wilkinson [4] studied dissipation due to the Landau–Zener mechanism in the quantum mechanic adiabatic regime. Austin and Wilkinson [6], and later other researchers [7,8] tried to extend the study of quantum dissipation beyond the quantum mechanic adiabatic regime. Both in [6] and in [7] a random matrix theory (RMT) model was studied. A different RMT model was studied by Mizutori and Aberg [9]. The applicability of perturbation theory (linear response) and semiclassical considerations for the study of quantum dissipation were explored by Cohen [8].

A systematic investigation of the interaction of a slow system with a fast one is possible with the help of multiple scale analysis. Under such conditions Ott demonstrated [10] that the phase space volume enclosed by the energy surface of the fast particle is an adiabatic invariant, namely its change is much slower than that of the fast particle Hamiltonian. It has been verified for various conditions that it is indeed an adiabatic invariant [11]. In this paper we study the behaviour of a slow particle that is coupled to a fast chaotic system. A model for such a system, that is quite general, and has been studied by Berry and Robbins (BR) [12] and by Jarzynski [13] in the framework of multiple scale analysis, is defined by the Hamiltonian

$$\mathcal{H} = \frac{1}{2M}\boldsymbol{P}^2 + h(\boldsymbol{R}, \boldsymbol{z}). \tag{1}$$

The phase space coordinates of the slow particle are $(\boldsymbol{P}, \boldsymbol{R})$ and its mass is M. For simplicity it is coupled only through its position to the fast system whose phase space coordinates are $\boldsymbol{z} \equiv (\boldsymbol{p}, \boldsymbol{r})$. The latter system has the property that if $\boldsymbol{R}$ is kept fixed it is fully chaotic. The crucial feature of the system we wish to study in this work is that it exhibits a wide separation of timescales—the evolution of the fast system, characterized by the timescale T_{fast}, is so rapid that it explores all of the phase space accessible to it energetically before the slow particle, characterized by the timescale T_{slow}, moves appreciably. The adiabaticity parameter is $\varepsilon \sim T_{\text{fast}}/T_{\text{slow}}$ or $\varepsilon \sim V_{\text{slow}}/V_{\text{fast}}$, where V_{slow} and V_{fast} are the characteristic velocities of the slow and fast particles. One way to realize this is to couple two particles with a mass ratio of $m/M = \varepsilon^2 \ll 1$, as one can see by rescaling the equations of motion. The dynamics generated by the Hamiltonian (1) with the approximation that the slow particle evolves under the influence of the *average* force exerted on it by the fast system, which can be treated as a system described by a slowly varying Hamiltonian, was studied by BR. A formalism where the fluctuations of this force were studied was developed by Wilkinson [5] and by Jarzynski [13]. In this work this force will be calculated in the framework of RMT. The dynamics of the slow particle resulting from this force or its fluctuations will not be studied in this paper. Hence, in this paper only the back-reaction force resulting from a controlled slow change of $\boldsymbol{R}$ is calculated. First the classical dynamics is outlined and later the quantum mechanical behaviour is summarized. In the case of (1) the average force is given by

$$\boldsymbol{F}(\tau_a) = -\int \mathrm{d}\boldsymbol{z}\ \rho(\boldsymbol{z}, \tau_a)\partial_{\boldsymbol{R}} h(\boldsymbol{z}, \boldsymbol{R}(\tau_a)) \tag{2}$$

where $\rho(\boldsymbol{z}, \tau_a)$ is a normalized probability density in the fast particle phase space, and the dependence of $\boldsymbol{R}$ on the time τ_a is externally determined. For the purposes of this paper we may assume $\dot{\boldsymbol{R}} = \boldsymbol{V} = \text{const}$. The probability density satisfies the Liouville equation

$$\varepsilon \frac{\partial}{\partial \tau_a}\rho(\boldsymbol{z}, \tau_a) = \{h(\boldsymbol{z}, \boldsymbol{R}(\tau_a)), \rho(\boldsymbol{z}, \tau_a)\}_{\boldsymbol{z}} \tag{3}$$

written in a way that emphasizes that the evolution of the fast system is indeed on a much

shorter timescale than the timescale on which the fast Hamiltonian changes. $\{\ \}_z$ denotes Poisson brackets taken with respect to $\boldsymbol{z}$. With the aid of the multiple scale expansion

$$\rho(\boldsymbol{z}, \tau_a) = \sum_{l=0}^{\infty} \varepsilon^l \rho_l(\boldsymbol{z}, \tau_a). \tag{4}$$

BR [12] were able to calculate the force acting on the slow particle up to first order in ε:

$$\boldsymbol{F} \approx \boldsymbol{F}_0 + \varepsilon \boldsymbol{F}_1. \tag{5}$$

To leading order, the force is given by the classical analogue of the Born–Oppenheimer force:

$$F_{0i}(\tau_a) = -\partial_{R_i} E(\boldsymbol{R}) \tag{6}$$

where $E(\boldsymbol{R})$ is chosen such that the phase space volume enclosed by the energy surface of the fast particle, $\Omega(E(\boldsymbol{R}), \boldsymbol{R})$, is constant. The leading correction to $\boldsymbol{F}_0$ includes a velocity dependent force $\boldsymbol{F}_1$. It consists of two different forces. The first force is geometric magnetism and it has been studied analytically in the systems under discussion by BR [12] and numerically by Berry and Sinclair [14]. The second force is related to deterministic friction. This force has been studied in [5, 12, 13, 15]. A central question that can be addressed at this point is under which conditions the slow particle feels friction due to the velocity-dependent force $\boldsymbol{F}_1$. In this paper this deterministic friction is studied. For this purpose we confine the study to a case where $\boldsymbol{R}$ is replaced by a scaler time-dependent parameter X. The force $\boldsymbol{F}_1$ is now

$$F_1(\tau_a) = -K\dot{X} \qquad K \equiv \Sigma^{-1}\partial_E[\Sigma(E, X)I(E, X)]_{E=E(X)} \tag{7}$$

where $\Sigma(E, X) \equiv \partial_E \Omega(E, X)$, and

$$I(E, X) = \int_0^{\infty} \mathrm{d}t' C(E, X; t'). \tag{8}$$

The correlation function of the fluctuating force is

$$C(E, X; t') \equiv \langle (\partial_X \tilde{h})_{t'} (\partial_X \tilde{h})_0 \rangle_{E,X} \tag{9}$$

where $\tilde{h}(\boldsymbol{z}, X) = h(\boldsymbol{z}, X) - E(X)$, while

$$\langle \cdots \rangle_{E,X} \equiv \Sigma^{-1}(E, X) \int \mathrm{d}\boldsymbol{z} \delta(E - h(\boldsymbol{z}, X)) \cdots \tag{10}$$

denotes the microcanonical average, and $(\partial_X \tilde{h})_t$ is the fluctuating part of the force at time t. In addition to the velocity-dependent force (7), at the same order in ε, there is a force that does not depend on velocity [16]. This force can be expressed as a gradient of a time-dependent potential, and is therefore a correction to the Born–Oppenheimer force (6). If $K > 0$, the force F_1 can be considered as deterministic friction resulting from the absorption of energy by the motion of the fast particle.

The behaviour when the fast system is quantum mechanical has also been studied by BR [12] to first order in ε. They found that only the geometric magnetism part of $\boldsymbol{F}_1$ takes a non-vanishing value in this limit. In the case we study here F_1 vanishes since $K = 0$, and there is no deterministic friction to this order. This difference between the quantum and classical results is a consequence of the discreteness of the quantum spectrum. The quantum correlation function corresponding to (9) is

$$C(n, t) = \sum_{m \neq n} |\langle n|\partial_X \hat{h}|m\rangle|^2 \cos\left[\frac{t}{\hbar}(E_n - E_m)\right] \tag{11}$$

and the infinite-time integral (8) over it vanishes [12]. In order to understand how the crossover between the classical and quantum behaviour occurs, it is instructive to calculate the integral of

the correlation function over a finite time. Following BR we assume in the calculation that the initial state is an eigenstate of the Hamiltonian, $|n\rangle$. It was verified by BR that their result holds also if the initial state is a mixture. Our calculation can also be extended to a mixture leading to the same results. If the initial state is a pure state but not an eigenstate of the Hamiltonian one can check that within the assumptions of the paper the results are similar to the ones found if the initial state is an eigenstate of the Hamiltonian (see the appendix). The finite-time integral corresponding to (8) that should be calculated then is

$$I(t) = \int_0^t C(t')\,\mathrm{d}t'. \tag{12}$$

The calculation of this integral in RMT will be the main subject of this paper. The correlation function and its integral may depend on the initial state n. This dependence has been suppressed in the notation for simplicity. In what follows, it will become clear that it is not important for the results of this paper. Taking the classical limit $\overline{h} \to 0$ for any finite t and then the limit $t \to \infty$ should result in a non-vanishing value of $I(\infty)$, while for any finite value of $\overline{h}$, $I(\infty)$ should vanish. The friction on the timescale t is often proportional to $I(t)$ as can easily be inferred from (7) and (8). The experimental meaning of this statement will be clarified in what follows. In order to understand the mechanism of this discordance, BR studied a model correlation function where the levels were equally spaced. They found that the function is periodic in time with period $t_p = \overline{h}/\Delta E$, where ΔE is the level spacing. Moreover, in the classical limit, which in their model corresponds to taking $t_p \to \infty$, $C(t)$ approaches the classical correlation function. For systems whose classical dynamics is chaotic, the energy levels are not equally spaced, but rather are distributed according to RMT [17]. The long-time behaviour of $I(t)$ is determined by the levels nearest to n, namely $n \pm 1$, as can be seen from (11). Therefore, one may expect behaviour different from the one found for the equally spaced spectrum. The natural question to ask is whether there is a characteristic timescale for the crossover between the quantum behaviour of the integral $I(t)$ and its classical behaviour. The most naive answer to this question is that the characteristic timescale is the Heisenberg time T_H, because it is the only timescale in the problem, and it is on this timescale that the quantum to classical crossover usually takes place. On the other hand, one can argue that there is no timescale for this crossover at all [18]. In RMT the probability for two consecutive levels to be separated by a distance s behaves like s^β for small spacings [19–22]. Consequently, $\langle I_\beta(t)\rangle \sim t^{-\beta}$ for long times, where here $\langle\cdots\rangle$ denotes the RMT ensemble average, and I_β is the integral (12) for some β. The answer given by the analysis presented in this paper is surprising. For the Gaussian orthogonal ensemble (GOE), for which $\beta = 1$, one indeed finds that $\langle I_\beta(t)\rangle$ decays like $1/t$, but for the Gaussian unitary ensemble (GUE), for which $\beta = 2$, one finds that it decays like a Gaussian with a characteristic time proportional to the Heisenberg time or vanishes after the Heisenberg time depending of the parametric dependence on X. The integral $I_\beta(t)$ was also calculated for other values of β. Why is the nature of the decay of $I_\beta(t)$ important? There is the quantum–classical discordance that has already been mentioned, and one would like to analyse the scale that is required to observe the crossover between the regimes. It is relevant for some experiments, that will be mentioned below. The power required to change X at rate $\dot{X}$ is $\mathrm{d}E/\mathrm{d}t = K\dot{X}^2$. On the timescale t it is determined by $I_\beta(t)$ of (12), that should replace $I(E, X)$ in (7). It vanishes in the $t \to \infty$ limit. The function $I_\beta(t)$ was found to be related to the variance of relaxation rates in some RMT models for this problem [23]. The time over which the correlation function decays should be compared with other timescales present in the specific system studied. One such timescale is $T_2 \sim \varepsilon^{-2}$, which is the timescale for the breakdown of the first order of the multiple scale analysis. Non-perturbative effects, such as Landau–Zener tunnelling, become important on a timescale of T_{LZ}. In realistic experiments there is also the

timescale for quantum decoherence T_ϕ. In order to observe the classical to quantum crossover discussed in this work $\langle I_\beta(t)\rangle$ should exhibit substantial decay for $t \ll \min(T_2, T_{LZ})$ and for $t \lesssim T_\phi$. For the results of this work to hold, adiabaticity should also hold in the quantum regime. The standard condition for this is $T_{\rm slow} \gg T_H$. However, requirements of this nature should be re-examined in view of the adiabatic theorem without gaps proven by Avron and Elgart [24].

The model discussed in this work is relevant for some experimental situations. The first proposed experimental system of this nature is related to the response of small metallic grains at low temperatures [25]. Another example is of a molecular beam prepared in a classical configuration, where initially many levels are substantially populated. The beam travels in a slowly varying field [26]. Consequently, the internal dynamics in the molecules is in a slowly varying potential. On short timescales the behaviour is classical, the integral of the correlation function is positive and energy is absorbed in the motion of the internal degrees of freedom. On longer timescales the integral of the correlation function decays to zero and one therefore realizes that actually no energy is absorbed by the molecules. The outcome of the experiment depends on its timescale, $T_{\rm exp}$, and on the timescale of decoherence, T_ϕ. The energy absorption by the internal degrees of freedom is proportional to $I_\beta(T_{\rm exp})$. This assumes $T_{\rm exp} < T_\phi$, otherwise $I_\beta(t)$ is truncated by T_ϕ in a more complicated way. Another example is of quantum dots where parameters are varied adiabatically, like in pumping experiments, but with dots that are closed, so that their spectrum is discrete [27].

In section 2 a specific RMT model is defined. For this model the ensemble average of the integral of the correlation function (12) is calculated analytically. It is demonstrated that most of the contribution for long times originates from the nearest-neighbour levels. In section 3 the integral of the correlation function (12), predicted by the nearest-neighbour level spacing distribution, is calculated for various distributions, including some that are not related to RMT models. In section 4 the results of this work are analysed and discussed.

2. Random matrix models

The main purpose of this paper is to study (11) and its finite-time integral (12) in the framework of RMT. The reason for this is that random matrices describe many characteristic properties of realistic quantum-chaotic systems [17, 28–31]. For simplicity our random matrices will depend on one external parameter X. For each random matrix we shall be able to calculate both $C(t)$ and $I(t)$.

2.1. A simple RMT model

We wish to construct a random matrix model for some of the levels of a system whose quantum Hamiltonian depends on some parameter. The N levels we wish to simulate by the random matrix lie within an energy strip of width $\delta E(N)$, that depends on N. Later on we shall be interested in studying the semiclassical limit. The meaning of taking this limit in the present context is to increase the density of levels in the δE-strip: in the classical limit the spectrum becomes continuous. We shall work with the well studied Gaussian ensembles [21, 22]. These are defined through four parameters: β which defines the symmetry of the random matrices, their dimension N, the mean value of their elements, and their variance (given through the parameter μ^2). All of these need to be chosen carefully in terms of parameters of the physical system, being simulated by the random matrix. The symmetry of the ensemble should be chosen to correspond to the real system. If the latter exhibits time reversal symmetry then the ensemble is the orthogonal one ($\beta = 1$). If the system does not exhibit this symmetry then

the ensemble is unitary ($\beta = 2$). The mean value of the matrix elements can be chosen to be zero, which corresponds to setting the ensemble average of the reference level $\langle E_n \rangle = 0$. The mean level density satisfies the semi-circle law [21, 22, 32]:

$$\overline{\rho}_x(x) \approx \begin{cases} \dfrac{2N}{\pi}\sqrt{1-x^2} & |x| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

for large N, where we have used the definition: $x \equiv E/\sqrt{4\beta\mu^2 N}$. In section 2.3 the relation between the parameters of the RMT model and the ones of the physical system will be discussed.

We shall use the Hamiltonian introduced by Austin and Wilkinson [6] and model a parameter-dependent system by the $N \times N$ random matrix:

$$H(X) = H_1 \cos X + H_2 \sin X \tag{14}$$

where $H_{1,2}$ are $N \times N$ random matrices from the same GOE or GUE ensemble. There are three advantages to working with $H(X)$: (a) it belongs to the same ensemble that $H_{1,2}$ belong to, (b) the derivatives of its matrix elements also belong to the same ensemble, (c) the matrices $H(X)$ and $\mathrm{d}H(X)/\mathrm{d}X$ are statistically independent. If we insert $H(X)$ and $\mathrm{d}H(X)/\mathrm{d}X$ into (11), and then perform the ensemble average, we obtain

$$C_\beta(t) = \left\langle \sum_{m \neq n} |(\mathrm{d}H(X)/\mathrm{d}X)_{n,m}|^2 \cos\left[\frac{t}{\hbar}(E_n - E_m)\right] \right\rangle \tag{15}$$

where $(\mathrm{d}H(X)/\mathrm{d}X)_{n,m} \equiv \langle n|\, \mathrm{d}\hat{H}(X)/\mathrm{d}X|m\rangle$ and $\langle \cdots \rangle$ denotes RMT ensemble averaging. The subscript β denotes the symmetry: $\beta = 1$ for GOE and $\beta = 2$ for GUE. The correlation function C_β and its finite-time integral I_β are ensemble averaged. The $\langle \cdots \rangle$ will be dropped from these quantities for notational simplicity.

The statistical independence of $\mathrm{d}H(X)/\mathrm{d}X$ and $H(X)$ implies

$$C_\beta(t) = \sum_{m \neq n} \langle |(\mathrm{d}H(X)/\mathrm{d}X)_{n,m}|^2 \rangle \left\langle \cos\left[\frac{t}{\hbar}(E_n - E_m)\right] \right\rangle \tag{16}$$

while the fact that $\mathrm{d}H(X)/\mathrm{d}X$ belongs to the same ensemble as $H(X)$ implies

$$\langle |(\mathrm{d}H(X)/\mathrm{d}X)_{n,m}|^2 \rangle = \langle |(H(X))_{n,m}|^2 \rangle = \beta\mu^2 \tag{17}$$

for $m \neq n$, leading to

$$C_\beta(t) = \beta\mu^2 \sum_{m \neq n} \left\langle \cos\left[\frac{t}{\hbar}(E_n - E_m)\right] \right\rangle. \tag{18}$$

We would like to make the connection between $C_\beta(t)/\beta\mu^2$ and the form factor

$$K(t) = \int \left[\frac{1}{\overline{\rho}^2(E)} \langle \rho(E + \epsilon/2\overline{\rho})\rho(E - \epsilon/2\overline{\rho}) \rangle - 1 \right] \mathrm{e}^{\mathrm{i}2\pi\epsilon\tau}\, \mathrm{d}\epsilon \tag{19}$$

where $\rho(E) = \sum_i \delta(E_i - E)$ is the density of states and $\overline{\rho}(E)$ is the smoothed density of states. The variable ϵ is the energy measured in units of the mean level spacing $1/\overline{\rho}(E)$ and $\tau = t/T_H$ is time in units of the Heisenberg time, $T_H = h\overline{\rho}(E)$.

Equation (18) can be written in the following form:

$$C_\beta(\tau)/\beta\mu^2 = \int \left[\frac{1}{\overline{\rho}^2(E)} \langle \rho(E + \epsilon/2\overline{\rho})\rho(E - \epsilon/2\overline{\rho}) \rangle - \delta(\epsilon) \right] \mathrm{e}^{\mathrm{i}2\pi\varepsilon\tau}\, \mathrm{d}\epsilon \tag{20}$$

where the $\delta(\epsilon)$ results from the omission of the term $m = n$ in the sum (18). Comparing the last equation with (19) one can see that

$$C_\beta(\tau)/\beta\mu^2 = K(\tau) + \delta(\tau) - 1. \tag{21}$$

In this work we are mainly interested in the time integral of the correlation function (12):

$$I_\beta(\tau)/\beta\mu^2 T_H = \int_0^\tau \mathrm{d}\tau' C_\beta(\tau')/\beta\mu^2 = \left[\tfrac{1}{2} - \int_0^\tau \mathrm{d}\tau'(1 - K(\tau'))\right]. \tag{22}$$

In the limit $\tau \to \infty$ the term in the square brackets is just $R_2(\epsilon = 0)$, the two-point spectral correlation function at zero energy separation. It vanishes as a result of level repulsion.

In order to perform actual calculations we make use of the well known form factor for GOE and GUE [22]. It is standard to define

$$b(\tau) = 1 - K(\tau). \tag{23}$$

For GOE it is given for example in Mehta's book (see [22] p 137):

$$b(\tau) = \begin{cases} 1 - 2\tau + \tau \ln[1 + 2\tau] & \tau \leqslant 1 \\ -1 + \tau \ln\left[\dfrac{2\tau + 1}{2\tau - 1}\right] & \tau \geqslant 1 \end{cases} \tag{24}$$

from which one obtains:

$$\frac{I_1(\tau)}{\mu^2 T_H} = \begin{cases} \frac{1}{2} - [\frac{5}{4}(\tau - \tau^2) + \frac{1}{2}(\tau^2 - \frac{1}{4}) \ln[1 + 2\tau]] & \tau \leqslant 1 \\ \dfrac{1}{2} - \left[\dfrac{1}{2}(1 - \tau) + \dfrac{1}{2}\left(\tau^2 - \dfrac{1}{4}\right) \ln\left[\dfrac{2\tau + 1}{2\tau - 1}\right]\right] & \tau \geqslant 1. \end{cases} \tag{25}$$

For $\tau \to \infty$, $I_1(\tau)$ falls off asymptotically as

$$\frac{I_1(\tau)}{\mu^2 T_H} \sim \frac{1}{12\tau}. \tag{26}$$

For GUE (see [22] p 95):

$$b(\tau) = \begin{cases} 1 - \tau & \tau \leqslant 1 \\ 0 & \tau \geqslant 1 \end{cases} \tag{27}$$

from which one obtains

$$\frac{I_2(\tau)}{2\mu^2 T_H} = \begin{cases} \dfrac{1}{2} - \left[\tau - \dfrac{\tau^2}{2}\right] & \tau \leqslant 1 \\ 0 & \tau \geqslant 1. \end{cases} \tag{28}$$

In order to compare the analytical results that hold in the infinite-N limit with results for finite N, ensembles of the $N \times N$ matrices H_1 and H_2 of (14), belonging to GOE or GUE were generated numerically. The integral of the correlation function I_β was then calculated numerically, by ensemble averaging. The results are presented in figures 1 and 2 for GOE and GUE respectively, and compared with (25) and (28). Units where $\beta\mu^2 T_H = 1$ were used (see section 2.3). The numerical errors in the figures were calculated according to

$$\Delta I_\beta = \frac{1}{N_{\mathrm{ens}}} \sqrt{\langle I_\beta^2 \rangle - \langle I_\beta \rangle^2}$$

where N_{ens} is the number of matrices used in the ensemble average.

For completeness the results for the Gaussian symplectic ensemble (GSE) (for which $\beta = 4$) are obtained with the help of (see [22] p 166):

$$b(\tau) = \begin{cases} 1 - \frac{1}{2}\tau + \frac{1}{4}\tau \ln \mid 1 - \tau \mid & \tau \leqslant 2 \\ 0 & \tau \geqslant 2. \end{cases} \tag{29}$$

Following the calculation performed for the other ensembles one finds

$$\frac{I_4(\tau)}{4\mu^2 T_H} = \begin{cases} \frac{1}{2} - [\frac{1}{16}(14\tau - 5\tau^2) + \frac{1}{8}(\tau^2 - 1) \ln|1 - \tau|] & \tau \leqslant 2 \\ 0 & \tau \geqslant 2. \end{cases} \tag{30}$$

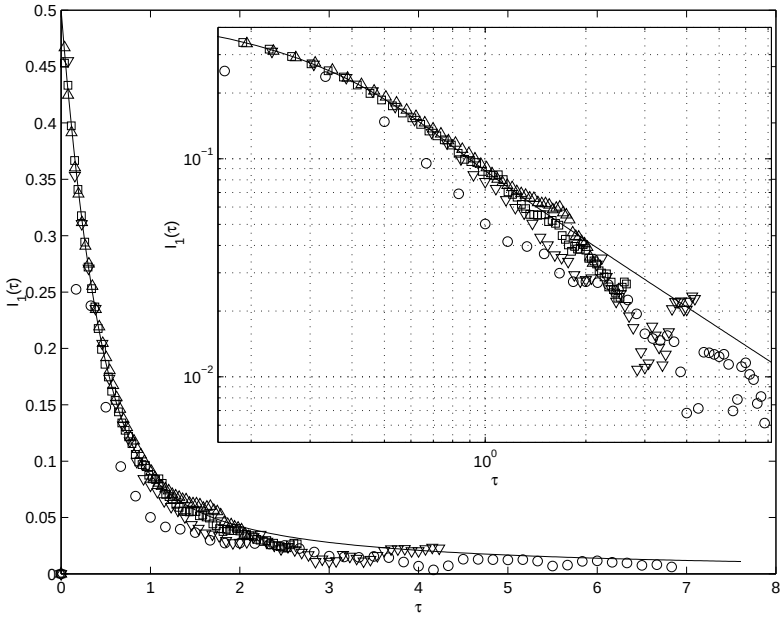

Figure 1. The integral of the correlation function for GOE. Numerical results for $N = 3$ (○), $N = 13$ (▽), $N = 53$ (□) and $N = 103$ (△) are shown. The curve is the large-N approximation (25). The inset shows the long-time behaviour on a log–log scale. The number of ensemble members used is 10^4 and the errors are of the order of $\Delta I_1 \approx 0.01$.

2.2. Nearest-neighbour level spacing dominance and the long-time limit

The model (14) is very specific in its dependence on the parameter X. An important property of this model is the statistical independence between $H(X)$ and $\mathrm{d}H(X)/\mathrm{d}X$. Such independence holds to a good approximation for disordered systems [33]. It is reasonable to also make this approximation for RMT models of chaotic systems. The reason is that most eigenstates look random, are statistically independent of the eigenvalues and therefore for many types of parametric dependences the matrix elements of $\mathrm{d}H(X)/\mathrm{d}X$ will look random and independent of the spectrum. Although this argument is reasonable for many types of parametric dependences it is clearly not general. For the asymptotic behaviour much less is required, since the long-time asymptotics is dominated by the nearest-neighbouring levels. The reason for this dominance is that if $\tau \gg 1$ the terms in the sum (15) oscillate wildly as a function of m, so that the important net contribution is from the terms nearest to being stationary. These are obviously $m = n \pm 1$. The approximation is therefore

$$C_\beta(t) \approx 2\left\langle |(\mathrm{d}H(X)/\mathrm{d}X)_{n,n-1}|^2 \cos\left[\frac{t}{\hbar}(E_n - E_{n-1})\right]\right\rangle \qquad \text{for} \quad t \gg T_H. \tag{31}$$

Only $m = n - 1$ is required since we know that the matrix elements and eigenvalue distributions are symmetric with respect to reflection around the middle eigenvalue n. We further approximate $|(\mathrm{d}H(X)/\mathrm{d}X)_{n,n-1}|^2$ by its mean value and ignore the contribution from its fluctuations resulting in

$$C_\beta(t) \approx 2\beta\mu^2\left\langle \cos\left[\frac{t}{\hbar}(E_n - E_{n-1})\right]\right\rangle. \tag{32}$$

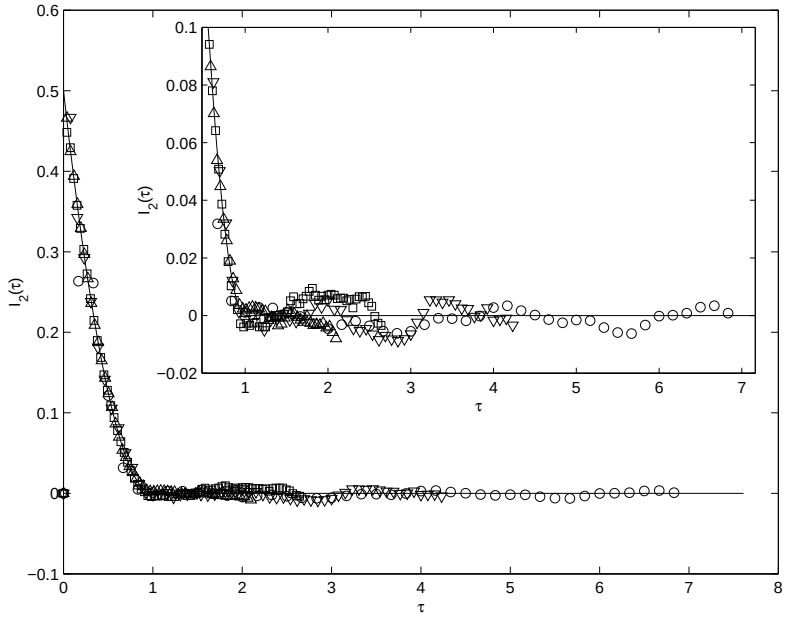

Figure 2. The integral of the correlation function for GUE. Numerical results for $N = 3$ (○), $N = 13$ (▽), $N = 53$ (□) and $N = 103$ (△) are shown. The curve is the large-N approximation (28). The number of ensemble members used is 10^4 and the errors are of the order of $\Delta I_2 \approx 0.01$. The inset shows a zoom-in on the long-time behaviour.

Hence we ignored

$$\Delta C_\beta(t) = 2\left\langle (|(\mathrm{d}H(X)/\mathrm{d}X)_{n,n-1}|^2 - \beta\mu^2)\cos\left[\frac{t}{\hbar}(E_n - E_{n-1})\right]\right\rangle \tag{33}$$

in the $t \gg T_H$ limit. In the framework of RMT (32) takes the form:

$$C_\beta(\tau)/\beta\mu^2 \approx 2\int_0^\infty \mathrm{d}s\, P_\beta(s)\cos(2\pi\tau s) \tag{34}$$

where s is the nearest-neighbour spacing in units of the mean level spacing $\Delta E = 1/\overline{\rho}(0)$, the time τ is measured in units of the Heisenberg time T_H and $P_\beta(s)$ is the probability distribution of s. The integral of the correlation function is

$$\frac{I_\beta(\tau)}{\beta\mu^2 T_H} \approx 2\int_0^\tau \mathrm{d}\tau' \int_0^\infty \mathrm{d}s\, P_\beta(s)\cos(2\pi\tau' s) = \frac{1}{\pi}\int_0^\infty \mathrm{d}s\,\frac{P_\beta(s)}{s}\sin(2\pi\tau s). \tag{35}$$

For the nearest-neighbour level spacing distribution we use the Wigner surmise [19], that is exact for 2×2 matrices, and takes the form (see equation (202) in [21]):

$$P_1(s) = \frac{\pi}{2}s\exp\left[-\frac{\pi}{4}s^2\right] \tag{36}$$

for GOE, and

$$P_2(s) = \frac{32}{\pi^2}s^2\exp\left[-\frac{4}{\pi}s^2\right] \tag{37}$$

for GUE. The integral (35) can be calculated for these distributions. For GOE one finds

$$\frac{I_1(\tau)}{\beta\mu^2 T_H} = \tfrac{1}{2}\exp\left[-4\pi\tau^2\right]\mathrm{erf}\left[\mathrm{i}2\sqrt{\pi}\tau\right]/\mathrm{i} \tag{38}$$

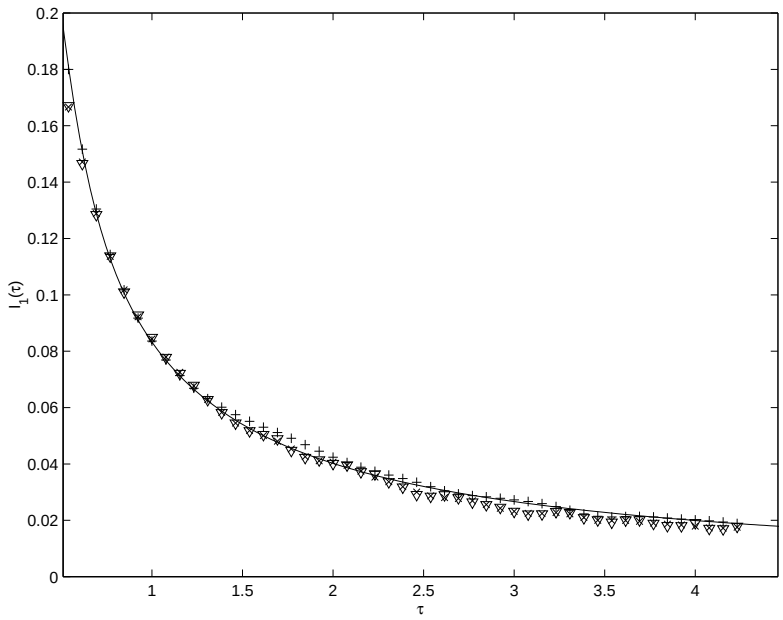

Figure 3. Testing the nearest-neighbour approximation (31) for GOE. Full numerical results for $N = 13$, including all level spacings ($\triangledown$), using only nearest and second nearest neighbours ($\times$) and using only nearest neighbours (+). Also shown is the long-time approximation (38), where only nearest-neighbour spacings are taken into account (curve). The number of ensemble members used is 10^5 and the errors are of the order of $\Delta I_1 \approx 0.005$.

while for GUE one finds

$$\frac{I_2(\tau)}{\beta\mu^2 T_H} = 2\tau \exp\left[-\frac{\pi^3}{4}\tau^2\right]. \tag{39}$$

A crucial approximation in the long-time regime is (31), where only the contribution of the nearest neighbours is taken into account. This approximation is tested in figures 3 and 4 for the model (14). In the numerical test the nearest and next-nearest-neighbour spacings are taken from the centre of the matrix. We see from these figures that the approximation (31) is quite reasonable and it improves as time increases. The reason is that as time grows the oscillations of the various terms in (15) with energy become stronger, enhancing the dominance of the nearest-neighbour contributions. The approximation is better for GOE than for GUE because the small s weight in the integral (35) is larger.

The asymptotic behaviour of erf [iy]/i for large real y is $\exp[y^2]/(\sqrt{\pi}\,y)$ [34]. Therefore, we find for GOE

$$\frac{I_1(\tau)}{\mu^2 T_H} \sim \frac{1}{4\pi\tau} \qquad \text{for} \quad \tau \gg 1 \tag{40}$$

and there is *no* characteristic time for the crossover from classical to quantum behaviour. This decay is extremely close to (26) hence for large τ the contribution of the nearest-neighbour spacings accounts for nearly all of $I_1(\tau)$.

For GUE the fall-off is much faster since in the large τ limit the Gaussian in (39) dominates, and there *is* a characteristic time $2/\pi^{3/2}$ (in units of the Heisenberg time), for the crossover to quantum behaviour. For long time $I_2(\tau)$ is extremely small, while it is exactly zero according to (28). Therefore, also for GUE (31) is an excellent approximation for the long-time limit.

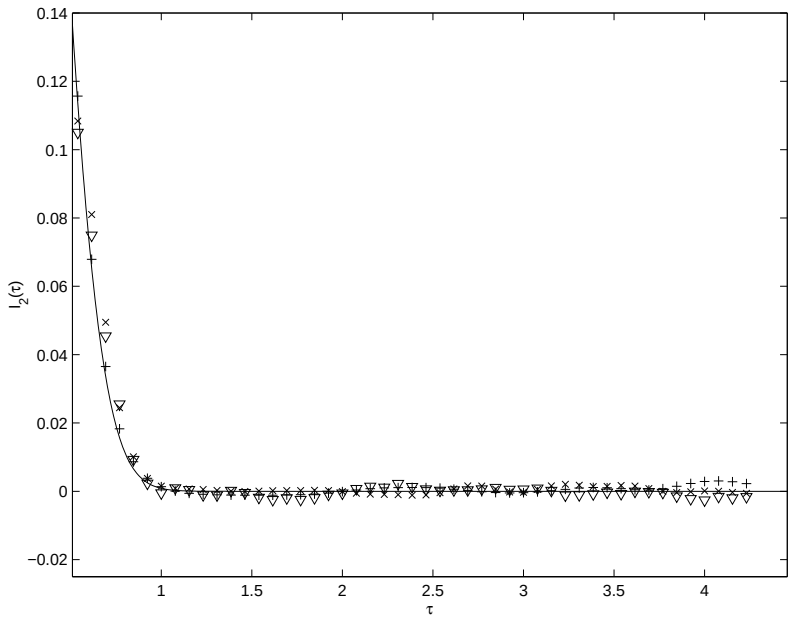

Figure 4. Same as figure 3, but for GUE, compared with the long-time approximation (39) (curve). The number of ensemble members used is 10^5 and the errors are of the order of $\Delta I_2 \approx 0.003$.

For completeness let us present the results for GSE, analogous to (38) and (39). For this we need to substitute the appropriate $P_\beta(s)$ into (35). According to (202) in [21]

$$P_4(s) = \frac{2^{18}}{3^6\pi^3} s^4 \exp\left[-\frac{64}{9\pi}s^2\right] \tag{41}$$

leading to

$$\frac{I_4(\tau)}{\beta\mu^2 T_H} = 2\left(1 - \frac{3\pi^3}{32}\tau^2\right)\tau \exp\left[-\frac{9\pi^3}{64}\tau^2\right]. \tag{42}$$

One can immediately see that there is a characteristic timescale for the quantum to classical crossover, as there was in the GUE case. As for the GUE case (42) is extremely small for long times, while (30) gives zero.

2.3. The short-time limit and determination of the parameters of the RMT model

In order to make contact with a physical system one has to relate the RMT parameters μ and N with $\overline{h}$ and the parameters of the physical system. This is done in the short-time limit. This limit is not universal and the dynamics of the chaotic system is *not* described by RMT. It is used only to determine the relation between the parameters. It will be assumed for concreteness that the system we wish to model by a random matrix is a two-dimensional chaotic billiard (a free particle of mass m in a two-dimensional box) of area $\mathcal{A}$. The results of the paper do not depend on this assumption. First we establish a relation between the mean density of states of this model to the one of RMT. In the framework of RMT the semicircle law (13) can be used for the mean density of states. If the density in the centre of the strip of energies, modelled by the random matrix, coincides with that of the two-dimensional billiard,

$$\overline{\rho}_x(0) = \frac{2N}{\pi} = \sqrt{4\beta\mu^2 N} \times \overline{\rho}_{2d}(E) = \sqrt{4\beta\mu^2 N} \times \frac{m\mathcal{A}}{2\pi\overline{h}^2}. \tag{43}$$

The existence of the semiclassical limit for the correlation function (11) [36] leads to another constraint. This constraint enables the expression of μ in terms of $\overline{h}$. It turns out that in the semiclassical limit the number of levels in a given interval grows with N. We shall give the explicit connection between classical parameters of the system, $\overline{h}$, μ and N in what follows. For the model (14) and other models where statistical independence between $\mathrm{d}H(X)/\mathrm{d}X$ and $H(X)$ holds, (18) can be used. In this framework it is easy to take the semiclassical limit. In this limit the spectrum can be considered continuous for fixed time, so that the sum can be replaced by a suitably weighted integral

$$C_\beta(t) \equiv \tilde{C}_\beta(t) - \beta\mu^2 \tag{44}$$

where

$$\tilde{C}_\beta(t) \approx \beta\mu^2 \int_0^\infty \mathrm{d}x \overline{\rho}_x(x) \cos\left[\frac{t}{\overline{h}}\sqrt{4\beta\mu^2 N}x\right] \qquad \text{for} \quad \overline{h} \to 0 \tag{45}$$

and $\beta\mu^2$ is the contribution of the $m = n$ term in (18). In the spirit of the RMT modelling we choose the eigenvalue E_n to be in the middle of the region described by RMT, therefore we set $E_n = 0$. The integral in (45) is known [35], and we obtain

$$\tilde{C}_\beta(t) \approx \sqrt{\beta\mu^2 N}\frac{\overline{h}}{t} J_1\left[2\sqrt{\beta\mu^2 N}\frac{t}{\overline{h}}\right] \qquad \text{for} \quad \overline{h} \to 0. \tag{46}$$

This correlation function has a characteristic timescale $T_c = \overline{h}/\sqrt{\beta\mu^2 N}$. Since the integral (12) for $I_\beta(t)$ is convergent, for all values of $t \gg T_c$ it is well approximated by its value at $t \gtrsim T_c$. Therefore, the infinite-time integral of the correlation function (46) is expected to take a classical value in the limit $N \to \infty$. One finds

$$\overline{I} = \overline{h}\sqrt{\beta\mu^2 N}. \tag{47}$$

Now units where $\mathcal{A}m/2 \equiv 2$ and $\overline{I} \equiv \frac{1}{2}$ are introduced. In such units μ and $\overline{h}$ are dimensionless and are given in terms of N as

$$\beta\mu^2 = \tfrac{1}{4}N^{-1/3} \tag{48}$$

$$\overline{h} = N^{-1/3}. \tag{49}$$

The Heisenberg time in these units is

$$T_H \equiv h\overline{\rho}_{2d}(E = 0) = 4N^{1/3} \tag{50}$$

and

$$\beta\mu^2 T_H = 1 \tag{51}$$

while the characteristic timescale for the saturation of the integral I_β to its classical value is $T_c = 2N^{-2/3}$. Finally, in these units

$$C_\beta(\tau) \approx \frac{J_1[4N\tau]}{8N^{1/3}\tau} - \frac{1}{4N^{1/3}} \qquad \text{for} \quad N \to \infty \tag{52}$$

where $\tau \equiv t/T_H$ is the dimensionless time. The integral over the correlation function is, in these units:

$$I_\beta(\tau) \approx \tfrac{1}{2} - \tau \tag{53}$$

and the approximation holds for $\tau \ll 1$. Now we can justify some of the assumptions that we made. First of all, we see that the limit $N \to \infty$ indeed corresponds to the limit $\overline{h} \to 0$. Finally, the mean level spacing $\Delta E = 1/\overline{\rho}_{2d}(0)$ and T_c decay to zero (as $N^{-2/3}$) in the limit $N \to \infty$, as expected.

3. The long-time behaviour predicted from the nearest-neighbour level spacing distribution

In the previous section the integral of the correlation function $I_\beta(\tau)$ was studied for RMT models. One conclusion was that it is dominated by the nearest-neighbour level spacings. It was found also that there is a big difference between GOE on the one hand and GUE and GSE on the other. In this section we shall assume nearest-neighbouring level dominance in order to study the decay of the correlation function if the nearest-neighbour level spacing distribution is given and will not rely on the assumption of an invariant RMT ensemble. We will also assume that the fluctuations of $|(\mathrm{d}H(X)/\mathrm{d}X)_{n,m}|^2$ are not important and that this quantity can be replaced by the constant $\beta\mu^2$. In absence of an invariant ensemble this may be a much cruder approximation. It will be assumed that the distribution of the nearest-neighbour level spacings is of the form

$$P_\beta(s) = cs^\beta \exp[-as^2] \tag{54}$$

where a and c are constants. The RMT distributions (36), (37) and (41) for $\beta = 1, 2, 4$ are of this form. The integral for the correlation function (35) takes the form

$$I_\beta(\tau)/\beta\mu^2 T_H = \frac{c}{\pi}\int_0^\infty \mathrm{d}s\, s^{\beta-1}\exp[-as^2]\sin(2\pi\tau s) = \frac{c}{\pi a^{\beta/2}}\mathcal{I}_\beta(y) \tag{55}$$

where

$$\mathcal{I}_\beta(y) \equiv \int_0^\infty \mathrm{d}s\, s^{\beta-1}\exp[-s^2]\sin sy \tag{56}$$

with $y = 2\pi\tau/\sqrt{a}$. The decay of this function will be explored in what follows, for arbitrary $\beta \geqslant 0$. One can verify that this function satisfies the ordinary differential equation

$$\begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}y^2}\mathcal{I}_\beta(y) + \frac{y}{2}\frac{\mathrm{d}}{\mathrm{d}y}\mathcal{I}_\beta(y) + \frac{\beta}{2}\mathcal{I}_\beta(y) = 0 \\ &\mathcal{I}_\beta(0) = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}y}\mathcal{I}_\beta(0) = \frac{1}{2}\Gamma\left[\frac{\beta+1}{2}\right]. \end{aligned} \tag{57}$$

Making the substitution

$$\mathcal{I}_\beta(y) = f_\beta(y/\sqrt{2})\exp[-y^2/8] \tag{58}$$

and changing to the variable $x = y/\sqrt{2}$, one arrives at a new differential equation

$$\begin{aligned} &f''(x) - [x^2/4 + 1/2 - \beta]f(x) = 0 \\ &f_\beta(0) = 0 \qquad f'_\beta(0) = \frac{1}{2}\Gamma\left[\frac{\beta+1}{2}\right]. \end{aligned} \tag{59}$$

Equation (59) is a well known equation, and its solutions are parabolic cylinder functions [34]: $\mathrm{U}[\frac{1}{2}-\beta, x]$, $\mathrm{V}[\frac{1}{2}-\beta, x]$. For arbitrary $\beta \geqslant 0$, the solution of (59) is a linear combination of these two functions:

$$f_\beta(x) = A_\beta U[\tfrac{1}{2}-\beta, x] + B_\beta V[\tfrac{1}{2}-\beta, x] \tag{60}$$

in which A_β, B_β are constants to be determined from the initial conditions. In particular, if β is an odd natural number it turns out that $A_\beta = 0$, if it is an even natural number $B_\beta = 0$ while for other $\beta \geqslant 0$ both A_β and B_β are non-vanishing. This fact is very important for the asymptotic behaviour of $\mathcal{I}_\beta(y)$, which is determined by the large-x behaviour of $U[\frac{1}{2}-\beta, x]$, $V[\frac{1}{2}-\beta, x]$ [34]:

$$U[\tfrac{1}{2}-\beta, x] \sim x^{\beta-1}\exp[-x^2/4] \tag{61}$$

$$V[\tfrac{1}{2}-\beta, x] \sim \sqrt{\frac{2}{\pi}}x^{-\beta}\exp[x^2/4] \tag{62}$$

as $x \to \infty$. One can immediately deduce that the first solution is subdominant for all β except for the special case when it is an even natural number, for which $B_\beta = 0$ in (60). Since $\mathcal{I}_\beta(y)$ is proportional to $I_\beta(\tau)$ and y is proportional to τ, for any fixed $\beta \neq 2n$ ($n = 1, 2, 3, \ldots$)

$$I_\beta(\tau) \sim \tau^{-\beta} \tag{63}$$

as $\tau \to \infty$, while for $\beta = 2n$ ($n = 1, 2, 3, \ldots$)

$$I_\beta(\tau) \sim \exp\left[-\frac{\pi^2}{a}\tau^2\right]\tau^{\beta-1} \tag{64}$$

as $\tau \to \infty$. This is precisely the type of behaviour found for the random matrix ensembles treated explicitly (39), (40) and (42).

Finally, one wonders what would the analogous results be in the case of the Poisson distribution. Unlike for the RMT systems, here there is a preferential basis. Since the correlation function is dominated by the nearest-neighbouring levels, and the corresponding eigenfunctions are typically localized far away from each other in phase space, one does not expect a systematic dependence of the matrix elements of $\mathrm{d}H/\mathrm{d}X$ on the level spacing. Therefore, the assumptions leading to (32) can be made here as well and returning to (31), one obtains for the correlation function

$$C_P(\tau) \approx 2\sigma_P^2 \int_0^\infty \mathrm{d}s\, P_P(s) \cos(2\pi\tau s) \tag{65}$$

where σ_P^2 is the variance of the off-diagonal matrix elements between nearest-neighbouring levels, which we shall leave unspecified, as we are only interested in the behaviour as a function of τ. For the Poisson case the nearest-neighbour spacing distribution is

$$P_P(s) = \exp[-s] \tag{66}$$

leading to

$$C_P(\tau) \approx \frac{2\sigma_P^2}{1 + (2\pi\tau)^2}. \tag{67}$$

The integral over this expression is

$$I_P(\tau) \approx \frac{\sigma_P^2 T_H}{\pi} \arctan(2\pi\tau) \tag{68}$$

the asymptotic behaviour of which is given by

$$I_P(\tau) \sim \frac{\sigma_P^2 T_H}{\pi}\left(\frac{\pi}{2} - \frac{1}{2\pi\tau} + \cdots\right) \to \sigma_P^2 T_H/2 \tag{69}$$

as $\tau \to \infty$. In the absence of level repulsion one indeed finds that the integral of the correlation function does not vanish. This does not result in any discordance with the classical limit where for an integrable system $I(t = \infty) = 0$ [12]. The reason is that for integrable systems the eigenfunctions of neighbouring energy levels typically have an exponentially small overlap (in $1/\hbar$). This small overlap is the physical reason for the Poisson distribution. Therefore, in the classical limit $\sigma_P^2 \to 0$ exponentially fast in $1/\hbar$.

Many systems that are neither integrable nor chaotic were found to have a semi-Poisson distribution [37]. For this case, that shows linear level repulsion, the nearest-neighbour spacing distribution is

$$P_{SP}(s) = 4s \exp[-2s]. \tag{70}$$

The integral of the correlation function is calculated along the lines of the calculation for the Poisson distribution. The result is

$$I_{SP}(\tau) \approx 2\sigma_{SP}^2 T_H \frac{\tau}{1 + (\pi\tau)^2} \tag{71}$$

which decays like $1/\tau$ in the long-time limit. For the semi-Poisson distribution there is level repulsion and indeed the integral of the correlation function decays with time.

4. Summary and discussion

The correlation function of the force applied by a fast quantum system on a slow classical one is calculated within the leading order correction to the adiabatic approximation following Berry–Robbins, Wilkinson and Jarzynski. Its finite-time integral $I(t)$ of (12) is proportional to the dissipation rate on the timescale t. In this work $I(t)$ was calculated under various statistical assumptions. In section 2 it was studied in the framework of RMT. For the specific dependence of the model (14) on the parameter X, the Hamiltonian H and $\mathrm{d}H/\mathrm{d}X$ are statistically independent [6]. For this case it was shown that up to a proportionality constant, the integral of the correlation function is simply related to the integral of the form factor (by (22)). Since the form factor is known in RMT, the integral of the correlation function was calculated and found to vanish for times beyond the Heisenberg time for GUE and to fall-off as a power law for GOE. This is a remarkable and surprising difference. The result for GSE is similar to the one found for GUE, except that the integral of the correlation function vanishes after twice the Heisenberg time. The properties of the model (14) are satisfied approximately by many systems [33], therefore it is expected that the results of this work are relevant for a wide range of problems. For the model (14) we have shown that for long times the results are dominated by the nearest-neighbour level spacings. If only these spacings are taken into account one finds that for long times the integral falls off as a power law for GOE and as a Gaussian, where the characteristic time is proportional to the Heisenberg time for GUE and GSE. These results do not require all the properties of the model (14). They require only that the contribution of the fluctuations in the absolute value of the matrix elements between nearest-neighbouring states (neighbouring in energy) is negligible, namely that the contribution of $\Delta C_\beta(t)$ of (33) can be ignored compared with that of $C_\beta(t)$. This is clearly a weaker assumption than complete statistical independence between H and $\mathrm{d}H/\mathrm{d}X$. Therefore, we expect the difference found between GOE and GUE (as well as GSE) to be generic for RMT models with various dependences on the parameter X. The long-time behaviour of the RMT models is expected to provide a faithful representation of the behaviour of chaotic systems, since it is dominated by the small level spacings. For short times, on the other hand, the behaviour depends on the specific properties of each system. The short-time behaviour of the RMT model was presented here only to set the relation between the constants of the RMT model and the ones of the chaotic system.

The assumption of the dominance of the contribution of nearest-neighbour level spacings, together with the assumption that $\Delta C_\beta(t)$ of (33) is negligible enables the calculation of the integral $I(t)$ of (12) for various distributions of nearest-neighbour level spacings even if these do not necessarily originate from RMT models. For the distribution (54), that is a generalization of the distributions found for GOE, GUE and GSE, one can calculate $I_\beta(t)$ as a function of β. In section 3 it is found to decay as $t^{-\beta}$ for all values of β, except when β is a positive even integer, for which it decays like a Gaussian with a characteristic time that is proportional to the Heisenberg time. What is special when β is a positive even integer? For these values the integral (55) can be extended to the range $(-\infty, \infty)$. The integrand is an entire function, the contour of integration can be deformed in the complex plane and the integral is dominated by a saddle point. For other values of β an extension of the integral to negative s, so that the integrand is analytic, is impossible. The point $s = 0$ is an end point and for large τ the integral is dominated by it, leading to power law decay. For non-integral β, the point $s = 0$ is also a singular point. It would be nice to find a more physical explanation for this difference between the various ensembles. For completeness the integral of the correlation function was calculated for the Poisson and the semi-Poisson distributions.

The various RMT formulae (25) and (28) are developed for the limit of infinite matrices.

This limit is approached extremely fast, as can be seen in figures 1 and 2.

The crucial approximation that was made generalizing the results beyond the model (14) was neglecting $\Delta C_\beta(t)$ of (33). Although reasonable, its validity for chaotic systems should be checked. The results may hold also for mixed systems if sticking to integrable regions does not take place on timescales relevant for the calculation. For chaotic systems corrections of order higher than the leading one, in the adiabatic approximation, may lead to different behaviour after some time $\min(T_2, T_{LZ})$. Taking $I_\beta(\tau)$ from RMT and $\Omega(E)$ of the realistic system in question, K of (7) is often approximately proportional to $I_\beta(\tau)$ with a positive proportionality constant. Therefore, the power transferred to the fast system is $\mathrm{d}E/\mathrm{d}\tau \approx \kappa I_\beta(\tau)\dot{X}^2$, where κ is (usually) a positive constant. For large τ the integral of the correlation function $I_\beta(\tau)$ may be negative (see (30) and (42) and note that for some values of β the prefactors in (63) and (64) are negative). Negative $I_\beta(\tau)$ implies energy transfer from the fast system to the driving system, rather than friction. Dephasing, as a result of the coupling to the environment will destroy the quantum correlations on a timescale T_ϕ, therefore the results of this work are relevant for times shorter than T_ϕ. On longer timescales the integral (12) is effectively cut off on the scale of T_ϕ. For $T_\phi \gg T_H$, much stronger energy absorption by the fast chaotic systems is expected if they satisfy time-reversal symmetry (GOE) than in the absence of this symmetry (GUE), a result that should be verified experimentally, for example for systems mentioned at the end of the introduction.

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Appendix. The correlation function of a superposition of energy states

Assume that the initial state is $|\psi\rangle = \sum_n a_n|n\rangle$, where $|n\rangle$ are the eigenstates of the (frozen) Hamiltonian, $\hat{h}|n\rangle = E_n|n\rangle$. In this case (11) is replaced by

$$C(\psi, t) = \sum_{m,n,n'} a^*_{n'} a_n \langle n'|\partial_X \tilde{h}|m\rangle\langle m|\partial_X \tilde{h}|n\rangle \times \tfrac{1}{2}[\mathrm{e}^{\mathrm{i}(E_{n'}-E_m)t/\hbar} + \mathrm{e}^{-\mathrm{i}(E_n-E_m)t/\hbar}] \tag{A.1}$$

where $\tilde{h}(X) \equiv \hat{h}(X) - E(X)$, in which $E(X) \equiv \sum_i |a_i|^2 E_i(X)$. Averaging over the ensemble of random matrices one finds the average correlation function

$$\langle C(\psi, t)\rangle = C(t) + C_0 \tag{A.2}$$

in which

$$C(t) = \sum_n \sum_{m\neq n} |a_n|^2 |\langle n|\partial_X \tilde{h}|m\rangle|^2 \cos\left[\frac{t}{\hbar}(E_n - E_m)\right] \tag{A.3}$$

that is a superposition of functions like (11). The time-independent part is given by

$$C_0 = \sum_m |a_m|^2 (\partial_X E_m - \partial_X E)^2 \tag{A.4}$$

which is the fluctuation in the force resulting from the fact that the initial distribution does not have a well defined energy. In such a situation (18) of [12] should be modified by adding a term, so that correlation function (9) vanishes in the $t \to \infty$ limit. This constant cancels the time-independent term C_0 and the correlation function reduces to (A.3).

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